Ultraharmonic Vibrations of Nonlinear Beams

J. A. BENNETT* AND R. L. RINKEL†
University of Illinois, Urban, Ill.

1. Introduction

RECENT experimental and theoretical investigations of the nonlinear vibrations of beams, where the nonlinearity is due to midsurface stretching, have shown that the ultraharmonics of the various modes may occur.[1,2] In particular the $\frac{3}{2}$, 2, and third-order ultraharmonics of the first mode have been observed for clamped-clamped beams. It is the purpose of this Note to investigate the behavior of the first mode ultraharmonics for various boundary conditions and beam properties.

The equation describing the moderately large deflections of a beam assuming the ends of the beam are restrained from axial motion may be written[1]

$$\rho h w_{,tt} + (EIw_{,xx})_{,xx} - \frac{Eh}{L}\left[v_0 + \frac{1}{2}\int_0^L (w_{,x})^2\,dx\right] w_{,xx} = P(x,t) \quad (1)$$

where L, h, and unity are the beam length, height, and width, respectively. This equation may be reduced to a set of nonlinear ordinary differential equations using Galerkins method if a solution is assumed in the form

$$w(x,t) = L\sum_{i=1}^{N} \xi_i(t)\phi_i(x) \quad (2)$$

where the $\phi_i(x)$ satisfy the boundary conditions. If the $\phi_i(x)$ are chosen as the linear mode shapes a wide variety of boundary conditions may be analyzed. For the present discussion only the fundamental mode will be considered. Numerical integration may be used to evaluate the integrals in the general case. If harmonic forcing is assumed Galerkin's method will reduce Eq. (1) to the following[1]

$$d^2\xi/d\bar{\tau}^2 + \omega_0^2\xi + G_1\xi^3 = F_0\cos\omega\bar{\tau} \quad (3)$$
$$\bar{\tau} = [(E/\rho)^{1/2}]\,t/L$$

The Duffing equation (3) has the well-known hard spring resonance curve. It is also known[2,3] that if a harmonic approximation to the solution of Eq. (3) is assumed this solution will be unstable in regions near $\omega/\omega_0 = 1/p = n/m$ where $n > m$, and n and m are relatively prime integers. In these regions an ultraharmonic instability of the type $\cos p\omega\tau$ arises in the solution. Although many of the instabilities are so weak that small amounts of damping will remove them, the lower order harmonics frequently occur.

2. Solutions in the Ultraharmonic Regions

It is convenient to generalize Eq. (3) further by making the further change of variable

$$\tau = \omega_0\bar{\tau}$$

Then

$$d^2\xi/d\tau^2 + \xi + G\xi^3 = P_0\cos(\omega/\omega_0)\tau \quad (4)$$

where

$$G = G_1/\omega_0^2; \qquad P_0 = F_0/\omega_0^2$$

If no ultraharmonics exist, the response curve is given by[1,2]

$$(\omega/\omega_0)^2 = 1 + GA_1^2 - P_0/A_1 \quad (5)$$

where ξ is approximated by

$$\xi = A_1\cos(\omega/\omega_0)\tau$$

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* Assistant Professor, Aeronautical and Astronautical Engineering Department. Member AIAA.

† Graduate Student; presently Captain USAF.

In the ultraharmonic region a solution of the form

$$\xi = A_1\cos(\omega/\omega_0)\tau + A_p\cos p(\omega/\omega_0)\tau \quad (6)$$

may be assumed. Equation (6) may be substituted into Eq. (4) and any harmonics which arise in this substitution which are not included in Eq. (6) will be neglected. This technique is called harmonic balance. The following two cases result:

Case 1)

$$p = 3$$
$$(\omega/\omega_0)^2 = 1 + G(\tfrac{3}{4}A_1^2 + \tfrac{3}{2}A_3^2 + \tfrac{3}{2}A_1A_3) - P_0/A_1 \quad (7)$$
$$9(\omega/\omega_0)^2 = 1 + G(\tfrac{3}{2}A_1^2 + \tfrac{3}{4}A_3^2)$$

Case 2)

$$p \neq 3$$
$$(\omega/\omega_0)^2 = 1 + G(\tfrac{3}{4}A_1^2 + \tfrac{3}{2}A_p^2) - p_0/A_1 \quad (8)$$
$$p^2(\omega/\omega_0)^2 = 1 + G(\tfrac{3}{2}A_1^2 + \tfrac{3}{4}A_p^2)$$

The third-order ultraharmonic [Eq. (7)] has been studied in some detail previously.[4] Therefore, attention will be concentrated on the more general case, Eq. (8). In obtaining these equations the lowest order harmonic that was neglected was of the order $\cos(2-p)(\omega/\omega_0)\tau$. If a term of this type is added to Eq. (6) the following three balanced equations are obtained. It is not practical to write a general set of equations applicable for all p's; therefore, only the special case $p = \frac{3}{2}$ is presented. A more general form that includes $p = \frac{3}{2}$, 2, 3, is given in[2]

$$(\omega/\omega_0)^2 = 1 + \tfrac{3}{2}(G/A_1)[A_1(\tfrac{1}{2}A_1^2 + A_{1/2}^2 + A_{3/2}^2) + A_1A_{1/2}A_{3/2}] - P_0/A_1 \quad (9a)$$

$$\tfrac{1}{4}(\omega/\omega_0)^2 = 1 + \tfrac{3}{2}(G/A_{1/2})[A_{1/2}(A_1^2 + \tfrac{1}{2}A_{1/2}^2 + A_{3/2}^2) + \tfrac{1}{2}A_{3/2}(A_1^2 + A_{1/2}^2)] \quad (9b)$$

$$\tfrac{9}{4}(\omega/\omega_0)^2 = 1 + \tfrac{3}{2}(G/A_{3/2})[A_{3/2}(A_1^2 + A_{1/2}^2 + \tfrac{1}{2}A_{3/2}^2) + \tfrac{1}{6}A_{1/2}^3 + \tfrac{1}{2}A_{1/2}A_1^2] \quad (9c)$$

Comparison of the solutions Eqs. (8) and (9) will indicate where the simpler solution (8) may be used. Equations (9) must be solved numerically while Eqs. (8) may be solved analytically. Solving Eq. (8) for A_p^2 and then ω/ω_0 we have

$$A_p^2 = (4/3G)[p^2(\omega/\omega_0)^2 - 1 - \tfrac{3}{2}GA_1^2]$$
$$(\omega/\omega_0)^2 = [1/(2p^2-1)][1 + \tfrac{9}{4}GA_1^2 + P_0/A_1] \quad (10)$$

Equations (10) have been plotted in Fig. 1 for the case of simply supported boundary conditions $h/L = 0.005$ and $p = \frac{3}{2}$. The curves for other ultraharmonics will be similar. Equations (9) have also been solved for the same values of the parameters and these results are also shown in Fig. 1. There are two points of interest: for sufficiently large values of the forcing function the ultraharmonic response curve is double valued; also, the simplified Eqs. (10) appear to be inaccurate for the double valued solution above the vertical tangent. These facts are of some physical significance. Consider a point (A) on the curve $F_0 = 10^{-7}$. As ω/ω_0 is increased the bifurcation point is reached and the ultra-

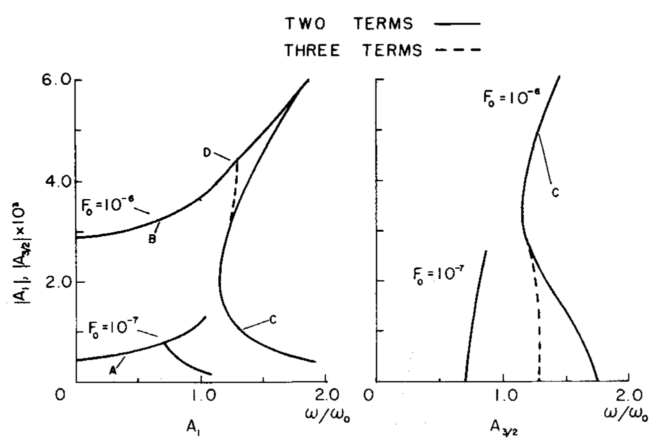

Fig. 1 Fundamental and $\frac{3}{2}$-order ultraharmonic response for a simply supported beam.

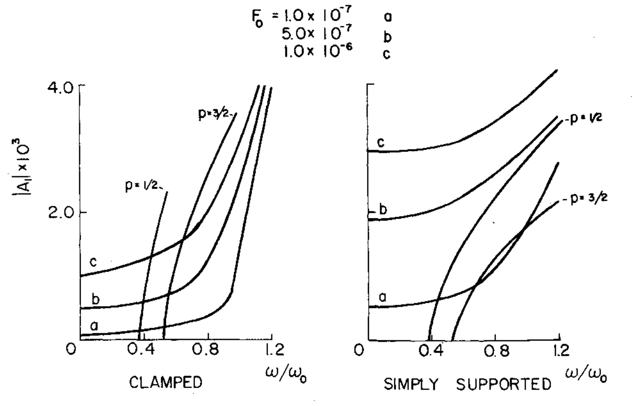

Fig. 2 Fundamental response and ultraharmonic vertical tangents for simply supported and clamped beams.

harmonic solutions will be followed until limited by damping. Now consider some point (B) on the curve $F_0 = 10^{-6}$. The bifurcation point (D) must be that given by the solutions to Eqs. (9); furthermore, when this point is reached, if the ultraharmonic is to occur the solution must jump to point (C). It is anticipated that unless a large perturbation occurs in the neighborhood of the bifurcation point that the ultraharmonic will not occur.

In order to substantiate this hypothesis Eq. (3) has been numerically integrated in the region just past the bifurcation point for $F_0 = 10^{-6}$ for several values of initial conditions. For values near the harmonic solution no tendency to jump to the lower branch of the curve was noted. The numerical integration has also confirmed the double valued aspects of the ultraharmonic response.

It is clear then that the location of the vertical tangent plays a significant part in determining the existence of the ultraharmonic response. If the vertical tangent of the $A_1 - \omega$ ultraharmonic solution lies below the fundamental response curve the ultraharmonic will be double valued and will probably not occur.

Figure 1 indicates that the vertical tangent may be adequately approximated by Eqs. (10). Rewriting the second of Eqs. (10) as

$$f = A_1(\omega/\omega_0)^2 - [A_1 + \tfrac{9}{4}GA_1{}^3 + P_0]/(2p^2 - 1)$$

The vertical tangency condition is

$$d(\omega/\omega_0)/dA_1 = 0 = -\partial f/\partial A_1/\partial f/\partial(\omega/\omega_0)$$

or

$$A_1{}^2 = [4(2p^2-1)/27G][1/(2p^2-1) - (\omega/\omega_0)^2] \qquad (11)$$

As shown previously, the ultraharmonic will not exist for P_0 greater than the value for which the line of vertical tangents passes through the bifurcation point, since for any greater P_0 the vertical tangent will be below the fundamental curve. Furthermore, the lowest frequency for the bifurcation point is $\omega/\omega_0 = 1/p$. The intersection of these points will determine a maximum P_0 for an ultraharmonic to occur for increasing ω/ω_0. Equation 5 may be solved for P_0, and A_1 replaced by Eq. (11). If the resulting equation is evaluated at $\omega/\omega_0 = 1/p$ the following is obtained

$$P_0 = [4(1-p^2)/27p^2]^{1/2}[8(p^2-1)/9p^2]\, G^{-1/2} \qquad (12)$$

Equation (12) will determine a maximum value of P_0 for an ultraharmonic to occur for increasing ω/ω_0.

The vertical tangent line (11) is quite dependent on the beam nonlinearity parameter G. The response curves (7) and the vertical tangent curves (11) have been plotted in Fig. 2 for several values of F_0 and for both simply supported and clamped boundary conditions, representing two extremes of support, again $h/L = 0.005$. If the vertical tangent lies below a response curve for a given F_0, in the region where the ultraharmonic can exist, the double valued ultraharmonic will exist. There is obviously a complex interplay among the various parameters which influence the behavior of the vertical tangent curve. It is clear however that any changes which increase the nonlinearity parameter G will flatten out the vertical tangent curve and lower the force level at which the double value phenomena will occur. The parameter G_1 which depends on integrals of the mode shape is less susceptible to change than the linear frequency ω_0. Therefore the ultraharmonic response can be changed most readily by increasing or decreasing the linear natural frequency. The boundary restraints play a significant role as shown in Fig. 2. For $F_0 = 5 \times 10^{-7}$ the $\frac{3}{2}$ ultraharmonic double valued ultraharmonic will occur for the simply supported boundary condition; whereas, it will not occur for the clamped boundary conditions.

3. Conclusions

It has been shown that for certain values of the parameters describing the geometry and forcing conditions of a nonlinear beam that the ultraharmonic response will change from a single valued to a double valued response curve. When the response curve is double valued it is concluded that for increasing ω/ω_0 the ultraharmonic response is extremely unlikely to occur. It has been demonstrated that for the double valued response the 3-term harmonic balance solution is necessary to predict the bifurcation point.

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A Consistent Treatment of Transverse Shear Deformations in Laminated Anisotropic Plates

E. REISSNER*
University of California, San Diego–La Jolla, Calif.

Introduction

RECENT interest in the analysis of the effect of transverse shear deformation in plates consisting of anisotropic laminations, and an apparent uncertainty about the appropriate way to extend results valid for homogeneous plates to the case of laminated plates[5] suggested to the writer that he should attempt to describe a way in which his earlier results for homogeneous isotropic plates,[1] and for sandwich plates with homogeneous cores,[2] can be generalized so as to apply to laminated anisotropic plates. For simplicity's sake this will be done in what follows for plates symmetric about their midplanes, without the previously considered possibility of coupled bending and stretching which exists without this symmetry.[4]

Our earlier results on the effect of transverse shear deformation were originally obtained through an application of the principle of minimum complementary energy in conjunction with the Lagrange multiplier method. It was subsequently shown that equivalent results could be obtained somewhat more simply

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* Professor of Applied Mechanics. Fellow AIAA.